

Some remarks on E -sequences in Noetherian rings

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In this short note we shall always assume that a ring A is a commutative Noetherian ring with unit element and A -modules are unitary. We use [2] as a reference source for homological algebra. Let \mathfrak{a} be an ideal of A and E be a finitely generated A -module with $\mathfrak{a}E \neq E$.

A sequence $\{a_1, \dots, a_k\}$ of elements in \mathfrak{a} is called an $\mathfrak{a}E$ sequence if

$$\mathfrak{a}_i E : (a_{i+1}) = \mathfrak{a}_i E, \text{ for } i=0, \dots, k-1,$$

where \mathfrak{a}_i is the ideal generated by a_1, \dots, a_i , and \mathfrak{a}_0 means the zero ideal of A . An $\mathfrak{a}E$ sequence $\{a_1, \dots, a_k\}$ is called a maximal $\mathfrak{a}E$ sequence if any sequence $\{a_1, \dots, a_k, a_{k+1}\}$ containing $\{a_1, \dots, a_k\}$ can not be an $\mathfrak{a}E$ sequence.

We shall prove in this note the following theorem:

Let A be a commutative Noetherian ring with an ideal \mathfrak{a} , and E be a finitely generated A -module. Then every $\mathfrak{a}E$ sequence can be extended to a maximal $\mathfrak{a}E$ sequence and all maximal $\mathfrak{a}E$ sequences have the same length.

Auslander and Buchsbaum [1], and Serre [6] have proved this theorem in the case of local ring with maximal ideal \mathfrak{m} by using the structure theorem of Cohen.

If a sequence $\{a_1, \dots, a_k\}$ is an $\mathfrak{a}E$ sequence, from the definition and $\mathfrak{a}E \neq E$ $\mathfrak{a}_i \subsetneq \mathfrak{a}_{i+1}$, ($i=0, \dots, k-1$). Hence the length of every $\mathfrak{a}E$ sequence is finite. The following arguments are modifications of the section 1 of [5].

LEMMA 1. *Let $\{a_1, \dots, a_i, a_{i+1}, a_{i+2}\}$ be an $\mathfrak{a}E$ sequence. Then $\{a_1, \dots, a_i, a_{i+2}, a_{i+1}\}$ is an $\mathfrak{a}E$ sequence if and only if*

$$(a_1, \dots, a_i)E : (a_{i+2}) = (a_1, \dots, a_i)E.$$

See the lemma 1.1 of [5].

LEMMA 2. *Let \mathfrak{a} and $(\mathfrak{a} \supseteq) \mathfrak{b}$ be two ideals of A and let a, a' be two elements of \mathfrak{a} which satisfy the conditions:*

$$\mathfrak{b} \cdot E : (a) = \mathfrak{b} \cdot E : (a') = \mathfrak{b} \cdot E.$$

Then the A/\mathfrak{a} -modules $\{(\mathfrak{b}, a)E : \mathfrak{a}\} / (\mathfrak{b}, a)E$ and $\{(\mathfrak{b}, a')E : \mathfrak{a}\} / (\mathfrak{b}, a')E$ are isomorphic.

Proof. By replacing E by $E/\mathfrak{b} \cdot E$, we can reduce the proof to the case $\mathfrak{b} \cdot E = (0)$, in which case a, a' are not zero-divisors in E . Let G be the set of elements in A which are not zero-divisors in E . We denote by E^* the module of quotients of E

with respect to G .

If we denote by $(E:\alpha)_{E^*}$ the subset of elements x of E^* which satisfy a condition $\alpha x \subseteq E$, $(E:\alpha)_{E^*}$ is an A -module. For any element x of $(E:\alpha)_{E^*}$, $\alpha x = z \in E$ and $\alpha z = \alpha \cdot \alpha x \subseteq \alpha E$, hence $z \in (\alpha E:\alpha)$. Therefore the mapping: $x \rightarrow z$ induces an isomorphism of $(E:\alpha)_{E^*}$ onto $(\alpha E:\alpha)$. From this isomorphism we obtain $(E:\alpha)_{E^*}/E \approx (\alpha E:\alpha)/\alpha E$. Similarly we can obtain an isomorphism $(E:\alpha)_{E^*}/E \approx (a'E):\alpha/a'E$, which proves the lemma.

The following property (a) is well known.

Let E be a finitely generated A -module. And \mathfrak{p}_i be the ideals belonging to a A -submodule F of E . For any ideal α of A , $(F:\alpha) = F$ if and only if α is not contained in any \mathfrak{p}_i .

LEMMA 3. *Let $\{a_1, a_2, \dots, a_k\}$, $\{b_1, b_2, \dots, b_k\}$ be two αE sequences of the same length. Then the A/α -modules $((a_1, \dots, a_k)E:\alpha)/(a_1, \dots, a_k)E$ and $((b_1, \dots, b_k)E:\alpha)/(b_1, \dots, b_k)E$ are isomorphic.*

We can prove the lemma similarly to [5], Th. 1.3 by Lemmas 1 and 2, and the above property (a).

Proof of Theorem. Let $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_t\}$ be two αE sequences. If $k < t$ we have an isomorphism

$$(*) \quad \begin{aligned} & ((a_1, \dots, a_k)E:\alpha)/(a_1, \dots, a_k)E \\ & \approx ((b_1, \dots, b_k)E:\alpha)/(b_1, \dots, b_k)E \end{aligned}$$

by Lemma 3. Since $\{b_1, \dots, b_k, b_{k+1}\}$ is an αE sequence, the right side of $(*)$ is equal to (0) . Hence $(a_1, \dots, a_k)E:\alpha = (a_1, \dots, a_k)E$. From the property (a), therefore, we can find an element a_{k+1} in α such that $\{a_1, \dots, a_k, a_{k+1}\}$ is an αE sequence. Thus we can obtain the theorem.

PROPOSITION 1. *Let A be a MC-ring¹⁾ and α be an ideal of A . Then the length of the maximal αA sequences is equal to the rank of α (in the sense of Krull).*

Proof. If the rank of α is equal to r , there exist elements $x_i (1 \leq i \leq r)$ in α such that $\text{rank}(x_1, \dots, x_i) = i (1 \leq i \leq r)$. Since A is a MC-ring, and $\text{rank } \alpha = \text{rank}(x_1, \dots, x_r)$, $\{x_1, \dots, x_r\}$ is a maximal αA sequence.

REMARK. We can obtain [6], Th. 2, [1], Prop. 3.4, and results of section 1 of [5], if we replace α by the maximal ideal \mathfrak{m} of a local ring, and E by A , respectively. From Proposition 1 we obtain [1], Prop. 2.7 and a fact that a MC-ring is a U -ring (see [4], p. 36).

The following lemma is stated in [3].

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- 1) A Noetherian ring A is called a MC-ring, (see [5]), if A satisfies a condition: if (a_1, \dots, a_r) is an ideal of rank r with a basis of r elements, then all prime ideals of (a_1, \dots, a_r) have rank r .

It is clear that a regular local ring is a MC-ring.

LEMMA 4. Let E be an A -module and $\{a_1, \dots, a_k\}$ be an \mathfrak{a} - E sequence. For any integer $n \geq 0$, we have

$$\mathrm{Tor}_m^A(A/\mathfrak{a}, E) = 0 \quad \text{for all } m > n$$

if and only if

$$\mathrm{Tor}_{m'}^A(A/\mathfrak{a}, E/\mathfrak{a}_k E) = 0 \quad \text{for all } m' > n + k.$$

In this case we have

$$\mathrm{Tor}_{n+k}^A(A/\mathfrak{a}, E/\mathfrak{a}_k E) \approx \mathrm{Tor}_n^A(A/\mathfrak{a}, E).$$

Hence we have $\dim_A A/\mathfrak{a} \geq k$, where \mathfrak{a}_k is an ideal generated by a_1, \dots, a_k .

We can easily obtain the lemma by the induction on k and the same method as that of the proof of [6], Prop. 3.

If $\{x_1, \dots, x_n\}$ is a maximal \mathfrak{a} - E sequence, then there exists some prime ideal \mathfrak{p} belonging to $(x_1, \dots, x_n)E$ such that $\mathfrak{p} \supseteq \mathfrak{a}$. Hence \mathfrak{p} contains some prime ideal \mathfrak{p}' belonging to (0) in E . We denote by $R_{\mathfrak{p}}$ the ring of quotients of R with respect to $R - \mathfrak{p}$. Then the functor $\otimes R_{\mathfrak{p}}$ is exact and $E \otimes R_{\mathfrak{p}} = E_{\mathfrak{p}}$ does not vanish from [2], VII, Exer. 10 and [1], Lemma 1.1. Since $\mathfrak{p} \supseteq \mathfrak{a}_{\mathfrak{p}}$, $\mathfrak{a}_{\mathfrak{p}} E_{\mathfrak{p}} \neq E_{\mathfrak{p}}$. By the same method as that of [1], Prop. 3.3 we can prove that $\{x_1, \dots, x_n\}$ is also a maximal $\mathfrak{a}_{\mathfrak{p}} E_{\mathfrak{p}}$ sequence. Hence we have

PROPOSITION 2. $\mathrm{codim}_{\mathfrak{a}} E = \mathrm{codim}_{\mathfrak{a}_{\mathfrak{p}}} E_{\mathfrak{p}} \leq \dim_{R_{\mathfrak{p}}} R_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}} \leq \dim_R R/\mathfrak{a}$ and $\mathrm{codim}_{\mathfrak{a}} E = \mathrm{codim}_{\mathfrak{a}_{\mathfrak{p}}} E_{\mathfrak{p}} \leq \mathrm{codim}_{\mathfrak{p}_{\mathfrak{p}}} E_{\mathfrak{p}} \leq \dim R_{\mathfrak{p}} \leq \dim R$, where \mathfrak{p} is some prime ideal such that $E_{\mathfrak{p}} \neq 0$, $\mathfrak{a}_{\mathfrak{p}} E_{\mathfrak{p}} \neq E_{\mathfrak{p}}$ and $\mathrm{codim}_{\mathfrak{a}} E$ is the length of maximal \mathfrak{a} - E sequences.

If A is a regular local ring and E is a finitely generated A -projective module, and if we replace \mathfrak{a} by the maximal ideal \mathfrak{m} in Proposition 2, then the equalities in Proposition 2 hold by [6]. Coro. 1 of Th.1.

Bibliography

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